1. Let G be a set, $e \in G$ and $*$ be a binary operator on G. When is $(G, *, e)$ called a group? Let $f : \mathbb{N} \to \mathbb{Z}$ be the bijective function given by $f(m) = m/2$ if m is even and $f(m) = (1-m)/2$ if m is odd. For $a,\ b \in \mathbb{N}$, let $a * b = f^{-1}(f(a) + f(b))$. Show that $(\mathbb{N}, *, 1)$ is an abelian group.
Solution:
We say $(G, *, e)$ is a group, if it satisfies the following properties
(i) Closure: for all $a,\ b \in G,\ a * b \in G$.
(ii) Associativity: for all $a,\ b,\ c \in G,\ a * (b * c) = (a * b) * c$.
(iii) Identity element: for every element $a \in G$, $e * a = a * e = a$.
(iv) Inverse element: for each $a \in G$, there exists an element $b \in G$ such that $a * b = b * a = e$.
As f is a bijection from $\mathbb{N}$ to $\mathbb{Z}$, for $a,\ b \in \mathbb{N},\ f^{-1}(f(a) + f(b)) \in \mathbb{N}$. So $a * b \in \mathbb{N}$. Let $a,\ b,\ c \in G$. Then we have $(f(a) + f(b)) + f(c) = f(a) + (f(b) + f(c))$, since $(\mathbb{Z}, +)$ is associative. Now

$$\begin{aligned} a * (b * c) &= a * (f^{-1}(f(b) + f(c))) \\ &= f^{-1}[f(a) + f(f^{-1}(f(b) + f(c)))] \\ &= f^{-1}[f(a) + (f(b) + f(c))] \\ &= f^{-1}[(f(a) + f(b)) + f(c)] \\ &= f^{-1}[f(f^{-1}(f(a) + f(b)))) + f(c)] \\ &= f^{-1}[f(a * b) + f(c)] \\ &= (a * b) * c. \end{aligned}$$

Therefore $(\mathbb{N}, *)$ is associative.

$1 \in \mathbb{N}$ and we have $f(1) = 0$. For any $a \in \mathbb{N}$, $a * 1 = f^{-1}[f(a) + f(1)] = f^{-1}(f(a)) = a$ and $1 * a = f^{-1}[f(1) + f(a)] = f^{-1}(f(a)) = a$. So 1 is unity. For $a \in \mathbb{N}$, $f(a) \in \mathbb{Z}$ so that $-f(a) \in \mathbb{Z}$. Therefore $f^{-1}(-f(a)) \in \mathbb{N}$. Let $b = f^{-1}(-f(a))$. Since $f^{-1}(0) = 1$, $a*b = f^{-1}[f(a)+f(b)] = f^{-1}[f(a)+(-f(a))] = f^{-1}(0) = 1$. So every element has Inverse in $\mathbb{N}$.

Let $a,\ b \in \mathbb{N}$. Then we have$f(a) + f(b) = f(b) + f(a)$, since $(\mathbb{Z}, +)$ is abelian. Consider $a * b = f^{-1}(f(a) + f(b)) = f^{-1}(f(b) + f(a)) = b * a$. Therefore $(\mathbb{N}, *)$ is an abelian group.
2. Let G be a group and $x \in G$. Define the order of x. Let $x,\ y \in G$ be of finite order. Show that the order of xy is finite, if G is abelian. Show that this fails if G is not abelian.
Solution:
If m is the least positive integer suchthat $x^m = e,\ e$ is the identity in G, we say the order of x is m.

Let $x, y \in G$ and the orders of $x,\ y$ are $m,\ n$ respectively.If G is abelian, $(xy)^i = x^i y^j$ for any $i \in \mathbb{Z}$. Now consider $(xy)^{mn} = x^{mn}y^{mn} = (x^m)^n(y^n)^m = e$. Therefore xy is of order less than mn.

If G is not abelian, the above is not true. For example, $G = GL_2(\mathbb{R}) = \{A_{2\times 2}(\mathbb{R}) : |A| \neq 0\}$ is a non abelian group under matrix multiplication. Let $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $o(A) = 2$, $o(B) = 2$ and $AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For any $n \in \mathbb{N}$, $(AB)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore the order of AB is infinite.
3. Show that the group $\mathbb{R}/\mathbb{Z}$ is isomorphic to $S^1 = \{z \in C : |z| = 1\}$. What is image of $\mathbb{Q}/\mathbb{Z}$ under the isomorphism you found in the previous part?.

Solution:
Define $\varphi : \mathbb{R}/\mathbb{Z} \mapsto S^1 = \{z \in C : |z| = 1\}$ by $\varphi(x + \mathbb{Z}) = e^{2\pi i x},\ 0 \leq x < 1$. Then φ is an isomorphism.

If φ is from $\mathbb{Q}/\mathbb{Z}$ to S^1. Then $\varphi(p/q + \mathbb{Z}) = e^{2\pi i p/q},\ 0 \leq p/q < 1$. Therefore $\varphi(\mathbb{Q}/\mathbb{Z}) = \{z \in \mathbb{C} : z^n =$

1, n th roots of unity, for some $n \in \mathbb{N}\}$.
4. Let G be a group. What is an automorphism of G?. Show the group of automorphisms of $(\mathbb{Z}/n, +)$ is isomorphic to $((\mathbb{Z}/n)^*, .)$. Compute $\mathrm{Aut}(\mathbb{Z})$.
Solution:
An automorphism φ from a group $(G, *)$ to itself is a bijective fuction such that $\varphi(a * b) = \varphi(a) * \varphi(b)$ for all $a,\ b \in G$.

To show $(\mathbb{Z}/n, +)$ is isomorphic to $((\mathbb{Z}/n)^*, .)$, see Theorem 6.4, Contemporary Abstract algebra by Joseph A. Gallian (page 125).

Let φ be an automorphism of $(\mathbb{Z}, +)$. Since 1 is a generator of $(\mathbb{Z}, +)$, $\varphi(1)$ is also a generator of $\varphi(\mathbb{Z}) = \mathbb{Z}$. So that $\varphi(1)$ is either 1 or -1. Therefore $\varphi_1(x) = x$, $\varphi_2(x) = -x$, for all $x \in G$, are the automorphisms of $(\mathbb{Z}, +)$.
5. When is a subgroup of a group G called normal subgroup? Let D_{14} be the dihedrel group of order 14. Show the only nontrivial proper normal subgroup of D_{14} is the subgroup consisting of rotations. Let $\phi : D_{14} \to S_5$ be a group homomorphism. Show that the image $\mathrm{Im}(\phi)$ has atmost two elements.
Solution:
Normal Subgroup: A subgroup H of G is said to be a normal subgroup of G if for every $g \in G$ and $h \in \mathbb{H},\ ghg^{-1} \in H$.

We know that set of all rotations in D_{14} is a subgroup of D_{14} and the order of this subgroup is 7. The number of 7-sylow's subgroups of D_{14} is $7k + 1$, where $7k + 1$ devides 14 and $k \in \{0\} \cup \mathbb{N}$. This gives $k = 0$, so that there are only one 7-sylow subgroup and it is of order 7. Therefore it is a normal subgroup of D_{14} and it is the subgroup consisting of rotations.

The number of 2-sylow's subgroups of D_{14} is $2k + 1$, where $2k + 1$ devides 14 and $k \in \{0\} \cup \mathbb{N}$. So number of 2-sylow's subgroups are either 1 or 7. The order of 2-sylow's subgroup is 2. If suppose D_{14} has unique 2-sylow subgroup, then it is normal subgroup of D_{14}. Now let H and K be the 7-sylow subgroup and 2-sylow subgroup respectively. As H and K are of prime order, they are cyclic groups. So $H =< x >,\ K =< y >$ for some $x,\ y \in G$. Since $H,\ K$ are normal subgroups of D_{14},we have $xy = yx$. So that $o(xy) = o(x)o(y) = 7.2 = 14$. This gives D_{14} is a cyclic group generated by xy, which is a contradiction. So D_{14} has 7 2-sylow subgroups. Therefore they are not normal subgroups. Hence D_{14} has unique proper normal subgroup.

Let $\phi : D_{14} \to S_5$ be a group homomorphism. Then $D_{14}/\ker(\phi) \cong \mathrm{Im}(\phi)$. As $\ker(\phi)$ is a normal subgroup of D_{14}, the possibilities of order of $\ker(\phi)$ are 1, 2, 7, 14. Since D_{14} has no normal subgroups of order 2, $|\ker(\phi)| \neq 2$. If $|\ker(\phi)| = 1$, then $|D_{14}|/|\ker(\phi)| = |\mathrm{Im}(\phi)| = 14$. This is not possible, since $\mathrm{Im}(\phi)$ is a subgroup of S_5 and 14 does not devide 120, order of S_5. So $|\ker(\phi)| \neq 1$. Therefore the order of $\ker(\phi)$ is either 7 or 14. If $|\ker(\phi)| = 7$, then $|\mathrm{Im}(\phi)| = |D_{14}|/|\ker(\phi)| = 2$. If $|\ker(\phi)| = 14$, then $|\mathrm{Im}(\phi)| = |D_{14}|/|\ker(\phi)| = 1$. Hence, in any case $\mathrm{Im}(\phi)$ has atmost two elements.

6. Define the centre of a group G. Supoose G has unique element x of order 2. Then show that x is in the centre of G.
Solution:
The centre of a group $G,\ Z(G) = \{x \in G | xy = yx, \forall y \in G\}$.

Let $x \in G$ be only the element of order 2. But, for any $y \in G,$ we have $o(y^{-1}xy) = o(x) = 2$. So that $y^{-1}xy = x$. Therefore $x \in Z(G)$.